SECTION 2.3: TECHNIQUES FOR COMPUTING LIMITS

Here, we introduce some mathematical power tools, the 'Limit Laws,' which enable us to compute limits analytically without any of the guesswork we experienced when trying to determine limits numerically or graphically.

Theorem (Basic Limits):

1. **Constant Rule:** $\lim\limits_{x\to a} b = b$ for any constant, b.
2. **Identity Rule:** $\lim\limits_{x\to a} x = a$

Theorem (Limits Respect Function Arithmetic):

Suppose $\lim\limits_{x\to a} f(x) = L$, $\lim\limits_{x\to a} g(x) = K$, and let k be any real number. Then:

1. **Constant Multiple Rule:** $\lim\limits_{x\to a} [kf(x)] = k\left[\lim\limits_{x\to a} f(x)\right] = kL$
2. **Sum / Difference Rule:** $\lim\limits_{x\to a} [f(x) \pm g(x)] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = L \pm K$
3. **Product Rule:** $\lim\limits_{x\to a} [f(x)g(x)] = \left[\lim\limits_{x\to a} f(x)\right]\left[\lim\limits_{x\to a} g(x)\right] = LK$
4. **Quotient Rule:** $\lim\limits_{x\to a} \left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{L}{K}$, provided $K \neq 0$.
5. **Power Rule:** $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = L^n$, where n is any natural number $(n = 1, 2, 3, \ldots)$
6. **Radical Rules:**
   - If n is **odd**, $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$.
   - If n is **even** and $L > 0$, $\lim\limits_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x\to a} f(x)} = \sqrt[n]{L}$.
   - If n is **even** and $L = 0$, $\lim\limits_{x\to a} \sqrt[n]{f(x)} = 0$ provided $f(x) \geq 0$ for all x near a.

   NOTE: Recall if n is **even**. $\sqrt[n]{x}$ is not a real number if $x < 0$. Hence the extra caution here.

EXAMPLE 1: Cite which limit properties are being used to justify each step of evaluating the following limit:

$$\lim_{x\to 3} \frac{x\sqrt{x+1}}{x^2+2x-4} = \frac{\lim_{x\to 3} x\sqrt{x+1}}{\lim_{x\to 3}(x^2+2x-4)}$$

Reason:

$$= \frac{(\lim_{x\to 3} x)\left(\lim_{x\to 3}\sqrt{x+1}\right)}{\lim_{x\to 3}(x^2) + \lim_{x\to 3}(2x) - \lim_{x\to 3} 4}$$

Reason: ________
Reason: ________

$$= \frac{3\sqrt{\lim_{x\to 3}(x+1)}}{(\lim_{x\to 3} x)^2 + 2\lim_{x\to 3} x - 4}$$

Reason: ________
Reason: ________

$$= \frac{3\sqrt{\lim_{x\to 3} x + \lim_{x\to 3} 1}}{(3)^2 + 2(3) - 4}$$

Reason: ________
Reason: ________

$$= \frac{3\sqrt{3+1}}{9+6-4}$$

Reason: ________
Reason: ________

$$= \frac{6}{11}$$

WHEW!

OBSERVATION: If we plug $x = 3$ into $\dfrac{x\sqrt{x+1}}{x^2+2x-4}$, we get the same value as $\lim\limits_{x\to 3} \dfrac{x\sqrt{x+1}}{x^2+2x-4}$!!!

DEFINITION: A function f is said to be **continuous** at $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

NOTE: If we think of $\lim_{x \to a} f(x)$ as being what we 'expect' to get from f as x **apporaches** a, and $f(a)$ as what we actually 'get' out of f **at** $x = a$, then to be continuous at $x = a$ means we get what we expect!

We'll have lots more to say about continuous functions later, but for now, rest assured that all of the functions you've studied in algebra (polynomials, rational functions, root and radical functions,[1] exponential functions, logarithm functions) and trigonometry (sines, cosines, tangents, etc.) are continuous **on their domains**. For now, we'll stick to using the essence of continuity to help us efficiently evaluate limits.

The Substitution Principle: If f is continuous at $x = a$, then to find $\lim_{x \to a} f(x)$, substitute $x = a$ and find $f(a)$.

EXAMPLE 2: (VIDEO) Use the substitution principle to evaluate the following limits.

1. $\lim_{x \to -1} \dfrac{2x^2 - 3x + 1}{\sqrt{x+5}}$ Ans: $\lim_{x \to -1} \dfrac{2x^2 - 3x + 1}{\sqrt{x+5}} = 3$

2. $\lim_{x \to 0} \dfrac{e^x + e^{-x}}{2}$ Ans: $\lim_{x \to 0} \dfrac{e^x + e^{-x}}{2} = 1$

3. $\lim_{x \to 1} (x \ln(x) - x)$ Ans: $\lim_{x \to 1} (x \ln(x) - x) = -1$

We need to exercise caution using the substitution principle when it comes to piecewise-defined functions and even-indexed radicals, as the next examples illustrate.

EXAMPLE 3: Let $f(x) = \begin{cases} 4 - x^2 & \text{if } x \leq 2 \\ 2x - 1 & \text{if } x > 2 \end{cases}$. Find the following, if they exist:

1. $f(2)$ Ans: $f(2) = 0$

2. $\lim_{x \to 2^-} f(x)$ Ans: $\lim_{x \to 2^-} f(x) = 0$

3. $\lim_{x \to 2^+} f(x)$ Ans: $\lim_{x \to 2^+} f(x) = 3$

4. $\lim_{x \to 2} f(x)$ Ans: $\lim_{x \to 2} f(x)$ does not exist.

QUESTION: Why **can't** we use direct substitution to determine $\lim_{x \to 3} \sqrt{3 - x}$? What about $\lim_{x \to 3^-} \sqrt{3 - x}$?

If $x > 3$, $\sqrt{3 - x}$ is not real, so $\lim_{x \to 3} \sqrt{3 - x}$ does not exist, so we can't use direction substitution.

We can use direct substitution to find $\lim_{x \to 3^-} \sqrt{3 - x}$ since we are restricting our inputs to $x < 3$ so $\sqrt{3 - x}$ is defned.

Moreover, $\lim_{x \to 3^-} \sqrt{3 - x} = \sqrt{3 - 3} = 0$

[1]... with the usual caveat about even-indexed roots ...

EXAMPLE 4: Consider: $\lim_{x \to 5} \frac{2x^2 - 9x - 5}{25 - x^2}$.

Substitution gives: $\frac{2(5)^2 - 9(5) - 5}{25 - (5)^2} = \frac{0}{0}$, which is an **indeterminate form**. However, notice that for $x \neq 5$,

$$\begin{aligned} \frac{2x^2 - 9x - 5}{25 - x^2} &= \frac{(x-5)(2x+1)}{(x-5)(-x-5)} \\ &= \frac{\cancel{(x-5)}(2x+1)}{\cancel{(x-5)}(-x-5)} \\ &= \frac{2x+1}{-x-5} \end{aligned}$$

Since $\lim_{x \to 5} \frac{2x^2 - 9x - 5}{25 - x^2}$ doesn't care what's happening **at** $x = 5$, only what's happening as $x \to 5$, we have:

$$\lim_{x \to 5} \frac{2x^2 - 9x - 5}{25 - x^2} = \lim_{x \to 5} \frac{2x+1}{-x-5} = -\frac{11}{10}.$$

The same reasoning can be generalized to the following theorem.

THEOREM: If $f(x) = g(x)$ for all x 'near' a, except possibly at $x = a$, and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} f(x) = L$.

ALGEBRAIC STRATEGIES FOR RESOLVING THE $\frac{0}{0}$ INDETERMINATE FORM:

- Factor numerator and denominator and cancel common factors.
- Simplify complex fractions.
- Multiply by conjugates.

EXAMPLE 5: Analytically[2] determine $\lim_{x \to 1^-} \frac{|x^2 - 1|}{x - 1}$.

Substitution gives: $\frac{|(1)^2 - 1|}{1 - 1} = \frac{0}{0}$, so we factor: $\lim_{x \to 1^-} \frac{|x^2 - 1|}{x - 1} = \lim_{x \to 1^-} \frac{|(x-1)(x+1)|}{x - 1} = \lim_{x \to 1^-} \frac{|x-1|\,|x+1|}{x - 1}$.

To help us simplify $\frac{|x-1|}{x-1}$, we recall from College Algebra that: $|u| = \begin{cases} -u & \text{if } u < 0 \\ u & \text{if } u \geq 0 \end{cases}$.

Note that as $x \to 1^-$, $x < 1$ so $x - 1 < 0$. Hence, $|x - 1| = -(x - 1)$, so we find the limit:

$$\lim_{x \to 1^-} \frac{|x^2 - 1|}{x - 1} = \ldots = \lim_{x \to 1^-} \frac{|x-1|\,|x+1|}{x - 1} = \lim_{x \to 1^-} \frac{-(x-1)\,|x+1|}{(x-1)} = \lim_{x \to 1^-} \frac{-\cancel{(x-1)}\,|x+1|}{\cancel{(x-1)}} = \lim_{x \to 1^-} -(x+1) = -2,$$

which matches with what we found in the previous section.

[2] We explored this limit numerically and graphically in the previous section.

EXAMPLE 6: Analytically determine: $\lim_{x\to 3} \dfrac{\dfrac{1}{x-1} - \dfrac{1}{2}}{x-3}$.

Substitution gives: $\dfrac{\dfrac{1}{3-1} - \dfrac{1}{2}}{3-3} = \dfrac{0}{0}$, an indeterminate form. Simplifying the complex fraction, we get:

$$\begin{aligned}
\lim_{x\to 3} \frac{\frac{1}{x-1} - \frac{1}{2}}{x-3} &= \lim_{x\to 3} \frac{\left(\frac{1}{x-1} - \frac{1}{2}\right)}{(x-3)} \cdot \frac{2(x-1)}{2(x-1)} \\
&= \lim_{x\to 3} \frac{2-(x-1)}{2(x-1)(x-3)} \\
&= \lim_{x\to 3} \frac{(3-x)}{2(x-1)(x-3)} \\
&= \lim_{x\to 3} \frac{-\cancel{(x-3)}}{2(x-1)\cancel{(x-3)}} \\
&= \lim_{x\to 3} \frac{-1}{2(x-1)}
\end{aligned}$$

Hence, $\lim_{x\to 3} \dfrac{\dfrac{1}{x-1} - \dfrac{1}{2}}{x-3} = \lim_{x\to 3} \dfrac{-1}{2(x-1)} = -\dfrac{1}{4}$.

EXAMPLE 7: Analytically determine: $\lim_{h\to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$.

Once again, direct substitution gives: $\dfrac{\sqrt{2+(0)} - \sqrt{2}}{(0)} = \dfrac{0}{0}$. We multiply by conjugates here:

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{2+h} - \sqrt{2}}{h} &= \lim_{h\to 0} \frac{\left(\sqrt{2+h} - \sqrt{2}\right)}{h} \cdot \frac{\left(\sqrt{2+h} + \sqrt{2}\right)}{\left(\sqrt{2+h} + \sqrt{2}\right)} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{2+h}\right)^2 - \left(\sqrt{2}\right)^2}{h\left(\sqrt{2+h} + \sqrt{2}\right)} \\
&= \lim_{h\to 0} \frac{(2+h) - 2}{h\left(\sqrt{2+h} + \sqrt{2}\right)} \\
&= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{2+h} + \sqrt{2}\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{2+h} + \sqrt{2}}
\end{aligned}$$

Hence, $\lim_{h\to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h} = \lim_{h\to 0} \dfrac{1}{\sqrt{2+h} + \sqrt{2}} = \dfrac{1}{2\sqrt{2}}$.

EXAMPLE 8 (VIDEO): Analytically determine the following limits.

1. $\lim\limits_{x\to -2} \dfrac{2x^3+3x^2-2x}{x^2-4}$ Ans: $\lim\limits_{x\to -2} \dfrac{2x^3+3x^2-2x}{x^2-4} = -\dfrac{5}{2}$

2. $\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h}$ Ans: $\lim\limits_{h\to 0} \dfrac{(3+h)^2-9}{h} = 6$

EXAMPLE 9 (VIDEO): Analytically determine the following limits.

1. $\lim\limits_{t\to 2} \dfrac{\frac{3}{t-1}-3}{t-2}$ Ans: $\lim\limits_{t\to 2} \dfrac{\frac{3}{t-1}-3}{t-2} = -3$

2. Assuming $x \neq 0$, find: $\lim\limits_{h\to 0} \dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$ Ans: $\lim\limits_{h\to 0} \dfrac{\frac{1}{x+h}-\frac{1}{x}}{h} = -\dfrac{1}{x^2}$

EXAMPLE 10 (VIDEO): Analytically determine the following limits.

1. $\lim\limits_{x\to 4} \dfrac{\sqrt{2x+1}-3}{x-4}$ Ans: $\lim\limits_{x\to 4} \dfrac{\sqrt{2x+1}-3}{x-4} = \dfrac{1}{3}$

2. Assuming $x > 0$, find: $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$ Ans: $\lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{1}{2\sqrt{x}}$

EXAMPLE 11: Consider: $\lim\limits_{x\to 1} \dfrac{x^2-1}{x^2-2x+1}$. Direct substitution gives the $\dfrac{0}{0}$ indeterminate form, so factor:

$$\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-1)} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-1)} = \lim_{x\to 1} \frac{x+1}{x-1}$$

Since a factor of $(x-1)$ remains in the denominator, direct substitution on the reduced fraction gives: $\dfrac{2}{0}$.

Note $\dfrac{2}{0}$ is **not** an indeterminate form since the fraction becomes unbounded as the denominator approaches 0.

We can reason (and confirm numerically and graphically!) that:

- as $x \to 1^-$, $\dfrac{x+1}{x-1} \approx \dfrac{2}{\text{very small }(-)} \to -\infty$
- as $x \to 1^+$, $\dfrac{x+1}{x-1} \approx \dfrac{2}{\text{very small }(+)} \to \infty$

In a later section, we'll formally write that $\lim\limits_{x\to 1^-} \dfrac{x+1}{x-1} = -\infty$ while $\lim\limits_{x\to 1^+} \dfrac{x+1}{x-1} = \infty$.

For now, we'll write that these limits 'do not exist.'

IN GENERAL: If direct substitution results in a fraction of the form:

- $\dfrac{0}{0}$, there is a **possibility** the limit exists. Do some algebra to rewrite the limit.
- $\dfrac{\#}{0}$ for some nonzero number #, then the limit is unbounded and does not exist.

THE SQUEEZE THEOREM AND TRIGONOMETRIC LIMITS

EXAMPLE 12: Consider $\lim_{t\to 0} \frac{\sin(t)}{t}$. Direct substitution produces the indeterminate form: $\frac{\sin(0)}{0} = \frac{0}{0}$.

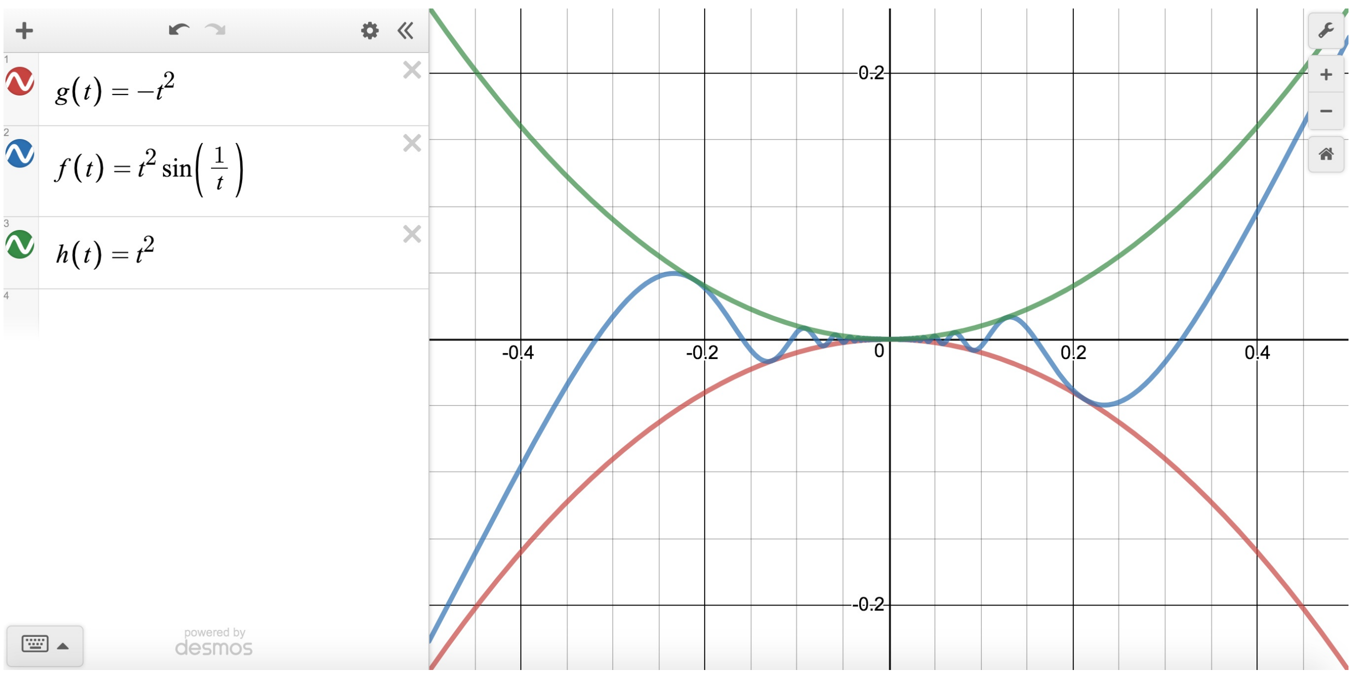
In the last section, we investigated this limit numerically and graphically and guessed that $\lim_{t\to 0} \frac{\sin(t)}{t} = 1$.

Since the fraction $\frac{\sin(t)}{t}$ doesn't simplify, we introduce another tool to help us prove our guess analytically:

THE SQUEEZE THEOREM: Suppose $g(x) \le f(x) \le h(x)$ for all x 'near' a, except possibly at $x = a$.
If $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$.

Geometrically, the inequality $g(x) \le f(x) \le h(x)$ means the graph of $y = f(x)$ is trapped between $y = g(x)$ and $y = h(x)$. If $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $y = f(x)$ is 'squeezed' to the same limit as g and h as $x \to a$.

Below is a typical 'Squeeze Theorem' scenario that will be used in the next example.



EXAMPLE 13 (VIDEO): Show $\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = 0$.

It is tempting to use the limit laws (incorrectly!) and write: $\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = \left(\lim_{t\to 0} t^2\right)\left(\lim_{t\to 0} \sin\left(\frac{1}{t}\right)\right) = 0 \cdot (\text{something}) = 0$.

While $\lim_{t\to 0} t^2 = 0$, we also know $\lim_{t\to 0} \sin\left(\frac{1}{t}\right)$ **does not exist**. This means **the limit laws do not apply**!

We can think of $t^2 \sin\left(\frac{1}{t}\right)$ as a sine wave with amplitude 't^2'. Even though $\sin\left(\frac{1}{t}\right)$ oscillates infinitely many times as $t \to 0$, the sine wave $t^2 \sin\left(\frac{1}{t}\right)$ is attenuated to a 0 amplitude, as seen in the graph above.

To formalize this observation, we note that like any sine wave of amplitude 1, $-1 \le \sin\left(\frac{1}{t}\right) \le 1$. Since $t^2 \ge 0$, we can multiply this inequality through by t^2 to obtain: $-t^2 \le t^2 \sin\left(\frac{1}{t}\right) \le t^2$. Since $\lim_{t\to 0}(-t^2) = \lim_{t\to 0} t^2 = 0$, the Squeeze Theorem gives $\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = 0$.

The Squeeze Theorem along with the Limit Laws can be used to prove the following:

LIMITS OF TRIGONOMETRIC FUNCTIONS:

- For all real numbers a, $\lim_{x\to a}\sin(x) = \sin(a)$ and $\lim_{x\to a}\cos(x) = \cos(a)$.
- For all real numbers a with $\cos(a) \neq 0$, $\lim_{x\to a}\tan(x) = \tan(a)$ and $\lim_{x\to a}\sec(x) = \sec(a)$.
- For all real numbers a with $\sin(a) \neq 0$, $\lim_{x\to a}\cot(x) = \cot(a)$ and $\lim_{x\to a}\csc(x) = \csc(a)$.
- $\displaystyle\lim_{t\to 0}\frac{\sin(t)}{t} = 1$ and $\displaystyle\lim_{t\to 0}\frac{1-\cos(t)}{t} = 0$.

NOTE: If a limit involving trigonometric functions results in the indeterminate form $\frac{0}{0}$, try teasing out either a factor of the form $\frac{\sin(t)}{t}$ and/or $\frac{1-\cos(t)}{t}$.

The proof of the above theorem is attached as an 'appendix.' It is worth your time to go through it at least once in your life. For now, it suffices to see the result in action in our next example.

EXAMPLE 14: Find each of the following limits analytically.

1. $\displaystyle\lim_{x\to\pi}\frac{\cos(x)}{1+\sin(x)}$

   Ans: $\displaystyle\lim_{x\to\pi}\frac{\cos(x)}{1+\sin(x)} = -1$

2. $\displaystyle\lim_{x\to 0}\frac{\tan(x)}{x}$

   Ans: $\displaystyle\lim_{x\to 0}\frac{\tan(x)}{x} = 1$

3. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x}$

   Ans: $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x} = 5$

4. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)}$

   Ans: $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{\sin(3x)} = \frac{5}{3}$

5. $\displaystyle\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

   HINT: Recall: $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$.

   Ans: $\displaystyle\lim_{h\to 0}\frac{\cos(\pi+h)+1}{h} = 0$

EXAMPLE 15 (VIDEO): Determine the following limits if they exist.

Check your answers numerically and graphically.

1. $\lim\limits_{x\to 0} \dfrac{\sin(117x)}{x}$ Ans: $\lim\limits_{x\to 0} \dfrac{\sin(117x)}{x} = 117$

2. $\lim\limits_{x\to 0} \dfrac{x}{\sin(117x)}$ Ans: $\lim\limits_{x\to 0} \dfrac{x}{\sin(117x)} = \dfrac{1}{117}$

3. $\lim\limits_{x\to 0} \dfrac{\sin(117x)}{\sin(42x)}$ Ans: $\lim\limits_{x\to 0} \dfrac{\sin(117x)}{\sin(42x)} = \dfrac{117}{42}$

4. $\lim\limits_{x\to 0} \dfrac{\tan(12x)}{x}$ Ans: $\lim\limits_{x\to 0} \dfrac{\tan(12x)}{x} = 12$

5. $\lim\limits_{x\to 0} x\csc(2x)$ Ans: $\lim\limits_{x\to 0} x\csc(2x) = \dfrac{1}{2}$

6. $\lim\limits_{x\to 0} \dfrac{1-\cos^2(x)}{x}$ Ans: $\lim\limits_{x\to 0} \dfrac{1-\cos^2(x)}{x} = 0$

7. $\lim\limits_{x\to 0} \dfrac{\sec(x)-1}{x}$ Ans: $\lim\limits_{x\to 0} \dfrac{\sec(x)-1}{x} = 0$

8. $\lim\limits_{x\to 0} \dfrac{x^2}{\sin(2x)\sin(3x)}$ Ans: $\lim\limits_{x\to 0} \dfrac{x^2}{\sin(2x)\sin(3x)} = \dfrac{1}{6}$

9. For $k \neq 0$, show that: $\lim\limits_{x\to 0} \dfrac{\sin(kx)}{x} = k$

10. For $k \neq 0$, show that: $\lim\limits_{x\to 0} \dfrac{\tan(kx)}{x} = k$

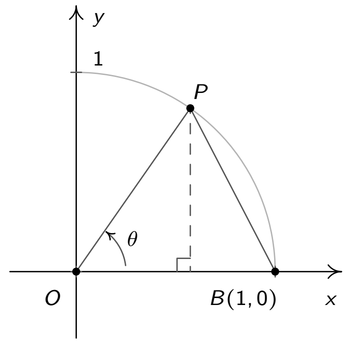
11. A function f is said to be **bounded** if there is a real number M so that $-M \le f(x) \le M$ for all x.

    Use the Squeeze Theorem to show that if f is bounded near 0 (except possibly at 0), then $\lim\limits_{x\to 0} x f(x) = 0$.

HOMEWORK: Section 2.3: 1 - 93 odd, 95*, 97*, 99*, 101*, 103*, Section 3.5: 11 - 21 odd

APPENDIX: PROVING THE TRIGONOMETRIC LIMITS (VIDEO)

1. Consider the portion of the Unit Circle below. Suppose θ is an acute angle graphed in standard position. Let P be the point where the terminal side of θ intersects the Unit Circle. This determines a triangle, $\triangle OPB$, and a circular sector, $\triangledown OPB$. Assume θ is measured in *radians*.



   (a) What are the coordinates of P in terms of θ? (Don't over think this!)

   (b) Show the area of $\triangle OPB$ is $\frac{1}{2}\sin(\theta)$.

   (c) Show the area of $\triangledown OPB$ is $\frac{1}{2}\theta$. **NOTE:** It's really important θ is measured in *radians* here!

   (d) By comparing the areas of $\triangle OPB$ and $\triangledown OPB$, show $0 \leq \sin(\theta) \leq \theta$.

   (e) Use your result from (d) and the Squeeze Theorem to show $\lim_{\theta\to 0^+} \sin(\theta) = 0$.

   (f) Use your result from (e) and the fact that sine function is *odd* to argue $\lim_{\theta\to 0^-} \sin(\theta) = 0$.

   (g) Use your results from (e) and (f) to conclude $\lim_{\theta\to 0} \sin(\theta) = 0$.

   (h) Use your result from (g) to show $\lim_{\theta\to 0} \cos(\theta) = 1$.

   HINT: For $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $\cos(\theta) = \sqrt{1-\sin^2(\theta)}$.

   (i) Use the sum-to-product formula:

$$\sin(x) - \sin(a) = 2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)$$

   along the Squeeze Theorem to show $\lim_{x\to a}[\sin(x) - \sin(a)] = 0$ and, hence $\lim_{x\to a}\sin(x) = \sin(a)$.

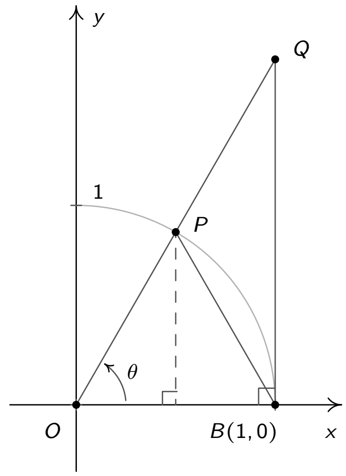
   (j) Use the sum-to-product formula:

$$\cos(x) - \cos(a) = -2\sin\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)$$

   along with your result from (i) to show $\lim_{x\to a}[\cos(x) - \cos(a)] = 0$ and, hence $\lim_{x\to a}\cos(x) = \cos(a)$.

NOTE: We now have enough to show that if T is any trigonometric function a is any number in the domain of T, $\lim_{x\to a} T(x) = T(a)$. In other words, the trigonometric functions are continuous on their domains.

2. Next, we extend a vertical line[3] from $(1, 0)$ until it intersects the terminal side of θ at the point Q.



(a) Find the coordinates of the point Q.

HINT: Use similar triangles!

(b) Show $\triangle OQB$ has area $\dfrac{1}{2}\tan(\theta)$.

(c) By comparing areas of $\triangle OPB$, $\triangledown OPB$, and $\triangle OQB$, show $\sin(\theta) \le \theta \le \tan(\theta)$.

(d) Use the inequality $\sin(\theta) \le \theta$ to show that $\dfrac{\sin(\theta)}{\theta} \le 1$.

(e) Use the inequality $\theta \le \tan(\theta)$ to show that $\cos(\theta) \le \dfrac{\sin(\theta)}{\theta}$.

(f) Put your results from (d) and (e) together to show $\cos(\theta) \le \dfrac{\sin(\theta)}{\theta} \le 1$.

(g) Use your result from part (f) along with the Squeeze Theorem to show $\displaystyle\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$.

(h) Show the function $f(\theta) = \dfrac{\sin(\theta)}{\theta}$ is *even* and use this and your result to part (g) to show $\displaystyle\lim_{\theta\to 0^-} \frac{\sin(\theta)}{\theta} = 1$.

(i) Use your results to parts (g) and (h) to get $\displaystyle\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$.

3. Show: $\displaystyle\lim_{\theta\to 0} \frac{1-\cos(\theta)}{\theta} = 0$.

HINT: Use a Pythagorean Conjugate!

[3] a line *tangent* to the Unit Circle